ISRAEL JOURNAL OF MATHEMATICS, Vol 27, Nos 3-4, 1977

A THEOREM ON LEVEL LINES OF CONTINUOUS FUNCTIONS

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ABSTRACT

A property of a continuous function $f(x)$, $x \in E_2$, similar to the classical intermediate value property is established. Namely, let a Jordan compact $J \subset E_2$ be the domain of definition of f. Then, for each parametrization $x(t)$, $0 \leqq t \leqq T$, $x(0) = x(T)$, of the boundary $\mathrm{Fr}(J)$ of J there exists a unique real λ and a unique connected component Φ of the level set $\{x \in J: f(x) = \lambda\}$ with the following property: any connected subset Ω of J containing "opposite" points of $\mathrm{Fr}(J)$ (i.e. points $x(t')$ and $x(t'')$ such that $t'' - t' = T/2$) has a common element with Φ.

THEOREM. *Suppose a function $f(x)$ is defined on a set J of the Euclidian plane E_2 and:*

1) *$f(x)$ is continuous on J;*

2) *J is compact and simply connected, and $L = \mathrm{Fr}(J)$ is a homeomorph of a circle, $t(0 \leqq t \leqq T)$ being one of the parameters realizing this homeomorphism. Then for some λ one of the components of the set $\{x \in J: f(x) = \lambda\}$ divides L into intervals such that the corresponding intervals of the parameter t do not exceed $T/2$. This λ and this component are unique. (In particular, if J has perimeter T, the parameter t being the arc length, then the length of each of the intervals in the theorem does not exceed $T/2$).*

For convenience two simple propositions are stated below as separate lemmas (for example, [1] and [2] contain enough to prove them).

LEMMA 1. *Let J be as in the theorem, the points p, p', q, q' lie on $L = \mathrm{Fr}(J)$ and are listed in their order on L. Let Ω and Ω' be two connected compact subsets of J, $p, q \in \Omega$, $p', q' \in \Omega'$. Then $\Omega \cap \Omega' \neq \varnothing$.*

LEMMA 2. *For a compact $\Omega \subset E_n$, let F be one of the components of the set*

Received December 19, 1975

$E_2\backslash\Omega$. Then the common boundary R of the sets Ω and F, $R = \Omega \cap \mathrm{cl}\, F$, is not empty and is compact and connected.

The following proposition will pave the way for the theorem's proof.

PROPOSITION. *Under the conditions of the theorem there exists for every $\Delta > 0$ such a set $\Omega \subset J$ that:*

a) *Ω is connected and closed;*

b) *the variation of f on Ω does not exceed Δ, i.e.*

$$\max\{f(x): x \in \Omega\} - \min\{f(x): x \in \Omega\} \leqq \Delta;$$

c) *no interval on L not containing points of Ω is such that the corresponding interval of t values is greater than $T/2$.*

PROOF. Let $z \in L$ and the numbers μ, ν be such that $\mu - \nu = \Delta$ and $f(z) \in [\nu, \mu]$. Let Ω_0 be the component of the set $J[\nu, \mu] = \{x \in J: f(x) \in [\nu, \mu]\}$ containing z. Then for Ω_0 the conditions a) and b) hold. Moreover, $\Omega_0 \cap L \neq \varnothing$. Choose a direction on L; for $x \in L$, denote by t_x the value of the parameter t corresponding to x, and for any two points $p, q \in L$ let $L(t_p, t_q)$ denote the piece of L between p and q in the chosen direction. The number $|t_q - t_p|$ will be called the scope of the arc $L(t_p, t_q)$. Consider intervals on L which do not contain Ω_0 points (the set of these intervals may be either finite or infinite). If none of these intervals have maximal scope, or if the maximal scope is less than or equal to $T/2$, then c) obviously holds. Therefore, let $L(t_p, t_q)$ be an interval such that $L(t_p, t_q) \cap \Omega_0 = \varnothing$ and $|t_p - t_q| > T/2$ (obviously, there can be only one such interval). Consider the component F of $J\backslash\Omega_0$ which contains the arc $L(t_p, t_q)$. Then, by Lemma 2, the common boundary R of the sets Ω_0 and F is connected. Moreover, $p, q \in R \subset \Omega_0$ (see Fig. 1).

Now note that the function $f(x)$ has a constant value f_0 on R, which equals either μ or ν. Indeed, assume $u \in R$ such that $\nu < f(u) < \mu$. Let $W(u, \delta)$ denote the component of $\{x \in J: |x - u| \leqq \delta\}$ that contains u. Since $f(x)$ is continuous, there exists such $\delta > 0$ that $W(u, \delta) \subset J[\nu, \mu]$. Since $W(u, \delta) \cap (J\backslash\Omega_0) \neq \varnothing$ while the set $\Omega_0 \cup W(u, \delta)$ is connected and lies in $J[\nu, \mu]$, there is a contradiction with Ω_0 being a component of $J[\nu, \mu]$.

Now let $\nu' = f_0 - \Delta/2$, $\mu' = f_0 + \Delta/2$ so that $\mu' - \nu' = \Delta$. Since the function $f(x)$ is uniformly continuous on J, there exists $\delta > 0$ such that $|x - x_0| \leqq \delta$, $x, x_0 \in J$ imply $|f(x) - f(x_0| \leqq \Delta/2$. The set $\Omega_0' = \bigcup_{x \in R} W(x, \delta)$ is connected and contained in $J[\nu', \mu']$. Let Ω_1 be the component of $J[\nu', \mu']$ such that $\Omega_0' \subset \Omega_1$. Obviously Ω_1 satisfies the conditions a) and b) (see Fig. 2). Since the parameter t

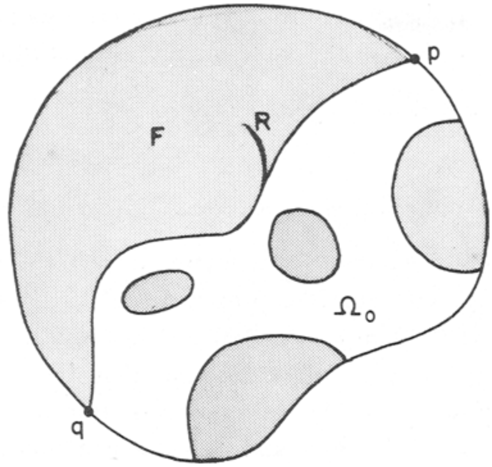

Fig. 1.

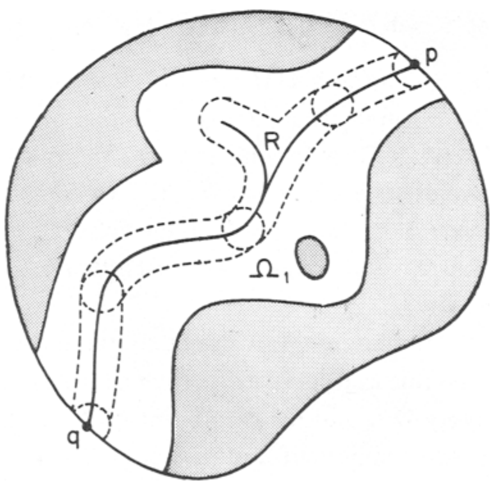

Fig. 2.

is a uniformly continuous function of point $x \in L$, there exists such $\varepsilon > 0$ that $x \in L$ implies $L(t_x - \varepsilon, t_x + \varepsilon) \subset W(x, \delta)$.

Consider intervals on L which do not contain Ω_1 points. Since $p, q \in \Omega_1$, any such interval must lie either on $L(t_p, t_q)$ or on $L(t_q, t_p)$. If any of these intervals has scope greater than $T/2$, it must lie in $L(t_p, t_q)$ and, moreover, in its subinterval $L(t_p + \varepsilon, t_q - \varepsilon)$ (assuming for instance $t_p < t_q$). Thus no interval not containing Ω_1 points can have a scope greater than $|t_p - t_q| - 2\varepsilon$.

Now construct in the same way Ω_2. Unless Ω_2 satisfies c), the only interval on L

not containing Ω_2 points whose scope exceeds $T/2$ has a scope less than or equal to $|t_p - t_q| - 4\varepsilon$. Proceeding in the same manner, in not more than

$$\left[\frac{|t_p - t_q| - T/2}{2\varepsilon}\right] + 1$$

steps, an Ω_k will be obtained which satisfies c) as well as a) and b).

Proof of Theorem. Consider a sequence $\Delta_i \to 0$ and the corresponding sequence of sets $\Omega^{(i)}$, satisfying the proposition. Let $\nu_i = \min\{f(x): x \in \Omega^{(i)}\}$, $\mu_i = \max\{f(x): x \in \Omega^{(i)}\}$. Then $\mu_i - \nu_i \leqq \Delta_i$. Let $\nu_i \leqq \lambda_i \leqq \mu_i$. The sequence $\{\lambda_i\}_1^\infty$ has a condensation point λ. It will be assumed that $\lim_{i\to\infty} \lambda_i = \lambda$.

Let Φ denote the set of condensation points of all sequences $\{x_i\}_1^\infty$ such that $x_i \in \Omega^{(i)}$. Obviously, $\Phi \subset J$, is closed and $f(x)$ is constant on Φ and equals λ. Let us prove that Φ is connected. Assume that the contrary is true: there exist sets U and V, open in E_2, such that

$$U \cap V = \emptyset, \qquad \Phi \subset U \cup V, \qquad \Phi \cap U \neq \emptyset, \qquad \Phi \cap V \neq \emptyset.$$

Denote $I_1 = \{i: U \cap \Omega^{(i)} \neq \emptyset,\ V \cap \Omega^{(i)} = \emptyset\}$. Since all the sets $\Omega^{(i)}$ are connected, it is easily verified that I_1 is infinite. Similarly the set $I_2 = \{i: V \cap \Omega^{(i)} \neq \emptyset,\ U \cap \Omega^{(i)} = \emptyset\}$ is infinite too. On the other hand, for any i, j, $\Omega^{(i)} \cap \Omega^{(j)} \neq \emptyset$, as there exists, for any pair $\Omega^{(i)}, \Omega^{(j)}$, a configuration of four points as in Lemma 1. But if $i \in I_1$ and $j \in I_2$, then $\Omega^{(i)} \cap \Omega^{(j)} \subset E_2 \setminus (U \cup V)$. This implies at once the existence of a point $z \in \Phi$ such that $z \notin U \cup V$, a contradiction. Thus Φ is connected.

Let us verify that L contains no intervals not containing Φ points whose scope is greater than $T/2$. Assume that there exists such an interval. Then it contains at least one point of every $\Omega^{(i)}$, and hence a point of Φ; a contradiction.

It is clear now that the component Σ of the set $\{x \in J: f(x) = \lambda\}$ containing Φ satisfies the theorem. The uniqueness follows immediately from Lemma 1.

Remarks.

1) The following gives an example of the theorem's application. Let J be a rectangle $a \leqq x \leqq b$, $a' \leqq y \leqq b'$. The theorem implies the existence of such a λ and a component Σ of the set $\{x \in J: f(x) = \lambda\}$, such that either every line $x = c$, $a \leqq c \leqq b$, or every line $y = c$, $a' \leqq c \leqq b'$, meets Σ.

2) The following example shows that the theorem cannot be strengthened so as to make Σ arcwise connected, even if $f(x)$ is infinitely differentiable.

Suppose J is the square $-2 \leqq x \leqq 2$, $-2 \leqq y \leqq 2$.

Let

$$f_1(x,y)=\begin{cases} e^{-1/x^2}\left(y-\sin\dfrac{1}{x}\right)^2 & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$$

and

$$f_2(x,y)=\begin{cases} e^{-1/(y-1)^2} & \text{for } 1<y\leqq 2\\ 0 & \text{for } -1\leqq y\leqq 1\\ e^{-1/(y+1)^2} & \text{for } -2\leqq y<-1.\end{cases}$$

The function $f=f_1+f_2$ is infinitely differentiable. The value $\lambda=0$ and the set

$$\Sigma=\left\{(x,y)\colon y=\sin\frac{1}{x},\ x\neq 0,\ -2\leqq x\leqq 2\right\}\cup\{(x,y)\colon x=0,\ -1\leqq y\leqq 1\}$$

satisfy the theorem for the perimeter as the parameter. Nevertheless Σ, which is unique, is not arcwise connected.

References

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